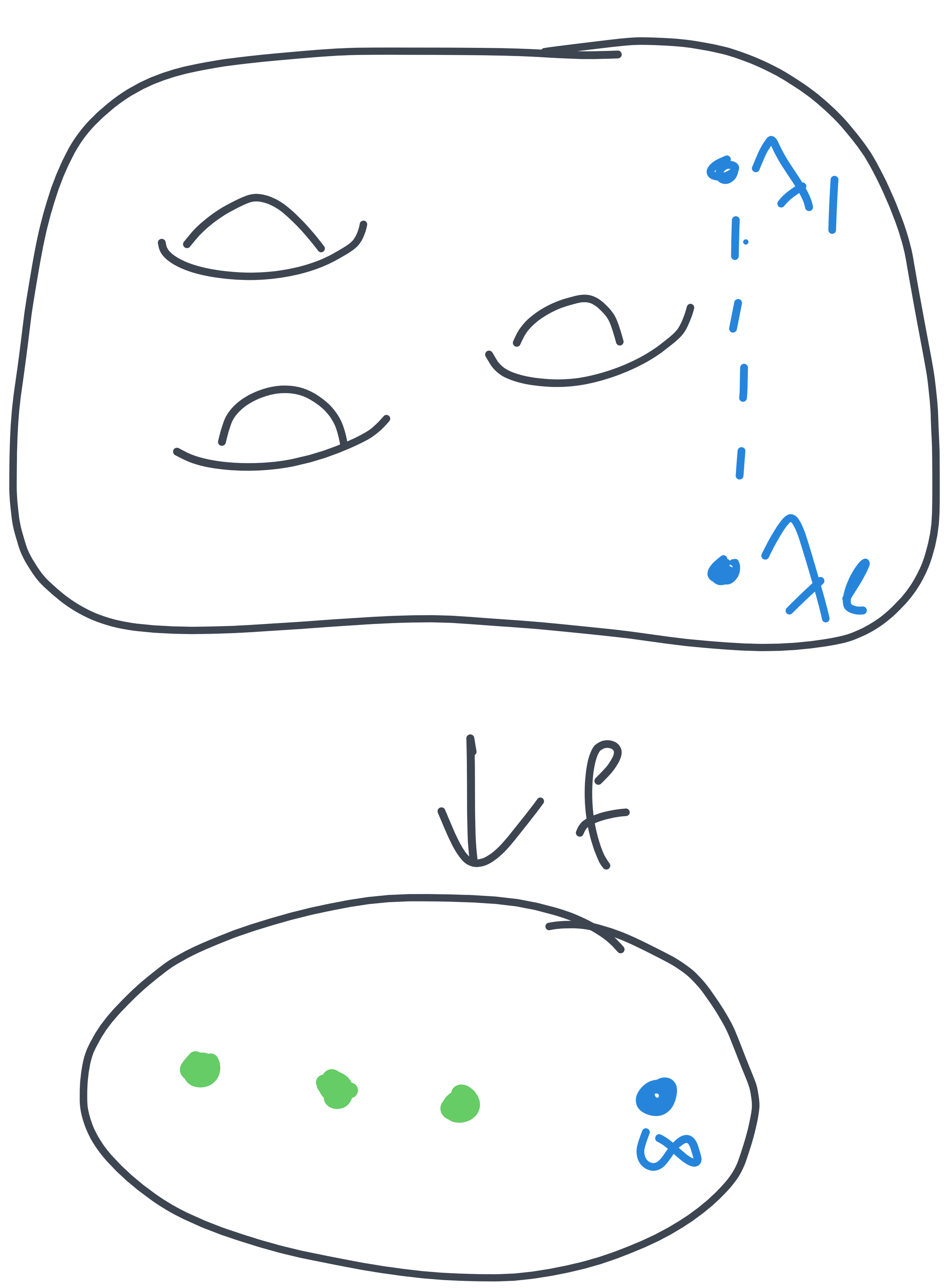
- Slides sent live to: "oliverleigh.org?ETH"
- Conventions: i) everything $/\mathbb{C}$
  ii) curves are connected
  iii) $g \in \mathbb{Z}_{\geq 0}$ and λ is an <u>ordered</u> partition
  iv) $\ell = \ell(\lambda)$, $d = |\lambda|$, $n = 2g-2+\ell+d$

Warm up: Hurwitz Numbers. Choose distinct $p_1, \dots, p_n \in \mathbb{P}^1 \setminus \{0\}$



$$H_{g;\lambda} = \# \left\{ \begin{array}{c} C \\ f \downarrow \\ \mathbb{P}^1 \end{array} \;\middle|\; \begin{array}{l} \bullet\ C \in \mathcal{M}_g \\ \bullet\ f^{-1}(\infty) = \sum_i \lambda_i q_i \text{ for some distinct } q_i \\ \bullet\ \text{branch divisor has} \\ \quad B_f - (d-\ell)(\infty) = p_1 + \cdots + p_n \end{array} \right\}$$

count w/ $\frac{1}{\mathrm{Aut}(f)}$

ELSV Formula $\quad n := 2g-2+\ell+d$

Theorem

$$H_{g,\lambda} = n!\left(\prod_i \frac{\lambda_i^{\lambda_i}}{\lambda_i!}\right)\int_{\overline{\mathcal{M}}_{g,\ell}} \frac{c(-\pi_*\omega_\pi)}{\prod(1-\lambda_i\psi_i)}$$

Proofs

i) [ELSV '99 & '01]

ii) [FP '02 & GV '03]

iii) [DKOSS '15]

① Construct a branch morphism

$$\mathrm{br} : \overline{\mathcal{M}}_g(\mathbb{P}^1,\lambda) \longrightarrow \mathrm{Sym}^n\mathbb{P}^1$$
$$[f] \longmapsto B_f - (d-\ell)(\infty)$$

② Show

$$H_{g,\lambda} = \int_{[\overline{\mathcal{M}}_g(\mathbb{P}^1,\lambda)]^{vir}} \mathrm{br}^*[p_1+\cdots+p_n]$$

③ Apply virtual localisation of [GP '99]

Hurwitz Relationships

One possible generalisation: Replace $\mathbb{P}^1$ with $\mathbb{P}^1[a]$. See JPT'11.

$$\int_{[\overline{\mathcal{M}}_g(\mathbb{P}^1,\lambda)]^{vir}} br^*[p_1+\cdots+p_n] \xleftrightarrow[\text{GV'03}]{\text{FP'02}} n!\left(\prod_i \frac{\lambda_i^{\lambda_i}}{\lambda_i!}\right)\int_{\overline{\mathcal{M}}_{g,\ell}} \frac{c(-\pi_*\omega_\pi)}{\prod_i(1-\lambda_i\psi_i)}$$

FP'02
GV'03

$H_{g,\lambda}$

ELSV'99
+ '01

DKOSS'15

Topological recursion

$$\int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,\lambda)]^{vir}} \prod_i \psi_i\, ev_i^*[p_i]$$

Pand.'00

H 1891

EMS'11

2x [OP'06]

Rep. theory of $Z(\mathbb{C}[S_d])$

Factorisations in S_d

"Burnside character formula" 1911

"r"-Hurwitz Relationships

$$\int_{[\overline{\mathcal{M}}_g(\mathbb{P}^1,\lambda)]^{vir}} br^*[p_1+\cdots+p_n] \; ?$$

$$= H_{g,\lambda}$$

$$r^{m+\ell+2g-2}\, m!\left(\prod_i \frac{\left(\frac{\lambda_i}{r}\right)^{\lfloor \frac{\lambda_i}{r}\rfloor}}{\lfloor \frac{\lambda_i}{r}\rfloor !}\right)\int_{\overline{\mathcal{M}}^{\frac{1}{r}}_{g,\ell}} \frac{c(-R\pi_*\mathcal{L})}{\prod_i\left(1-\frac{1}{r}\lambda_i\psi_i\right)}$$

[DKPS'19]

[BKLPS'17] : • g=0 • r=2

Topological recursion

Zu. '06

$$\int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,\lambda)]^{vir}} \prod_i \psi_i^r\, ev_i^*[p_i]$$

$$m = \frac{n}{r} = \frac{1}{r}(2g-2+\ell+d)$$

SSZ'15

[OP'06]

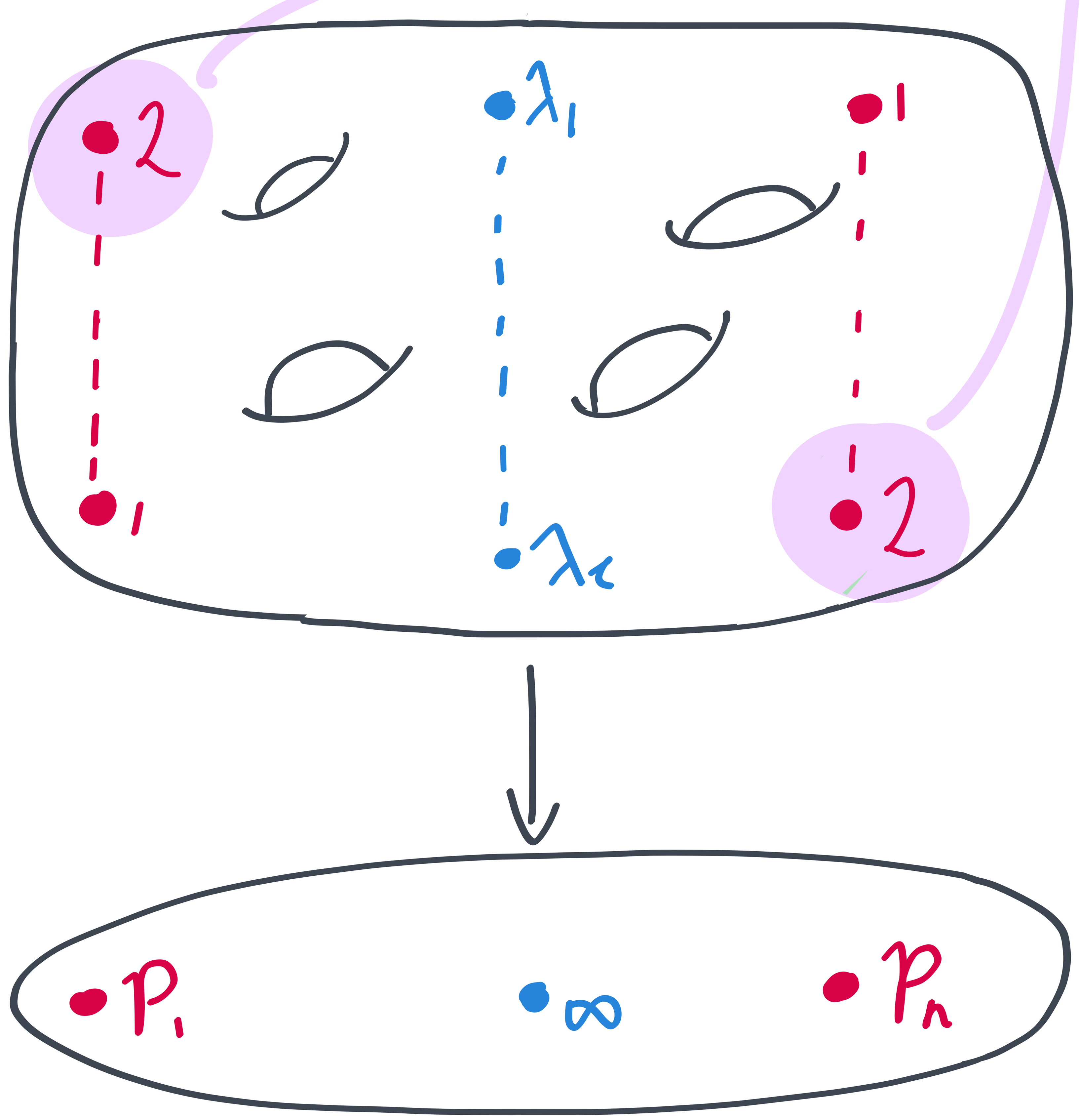
Rep. theory of $Z(\mathbb{C}[S_d])$

Factorisations in S_d

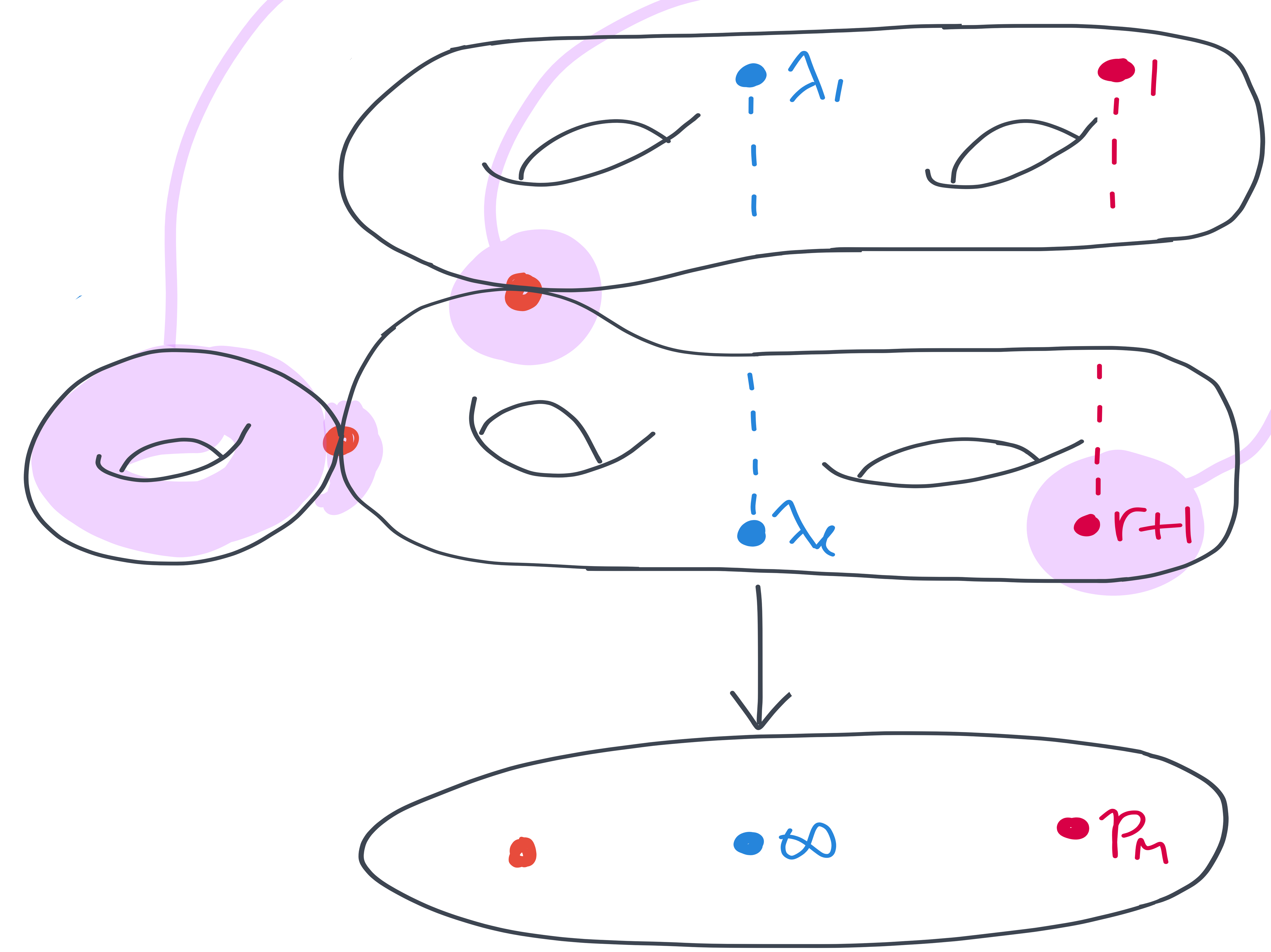
Qn What are we counting?

Ans Stable maps with conditions on the "conneted components of the non-étale locus" a.k.a. "special loci"

$r = 1$



$r > 1$



Ramification order of special loci

Determined by $br: \overline{\mathcal{M}}_g(\mathbb{P}^1,\lambda) \longrightarrow \mathrm{Sym}^n \mathbb{P}^1$ which is defined by a section:

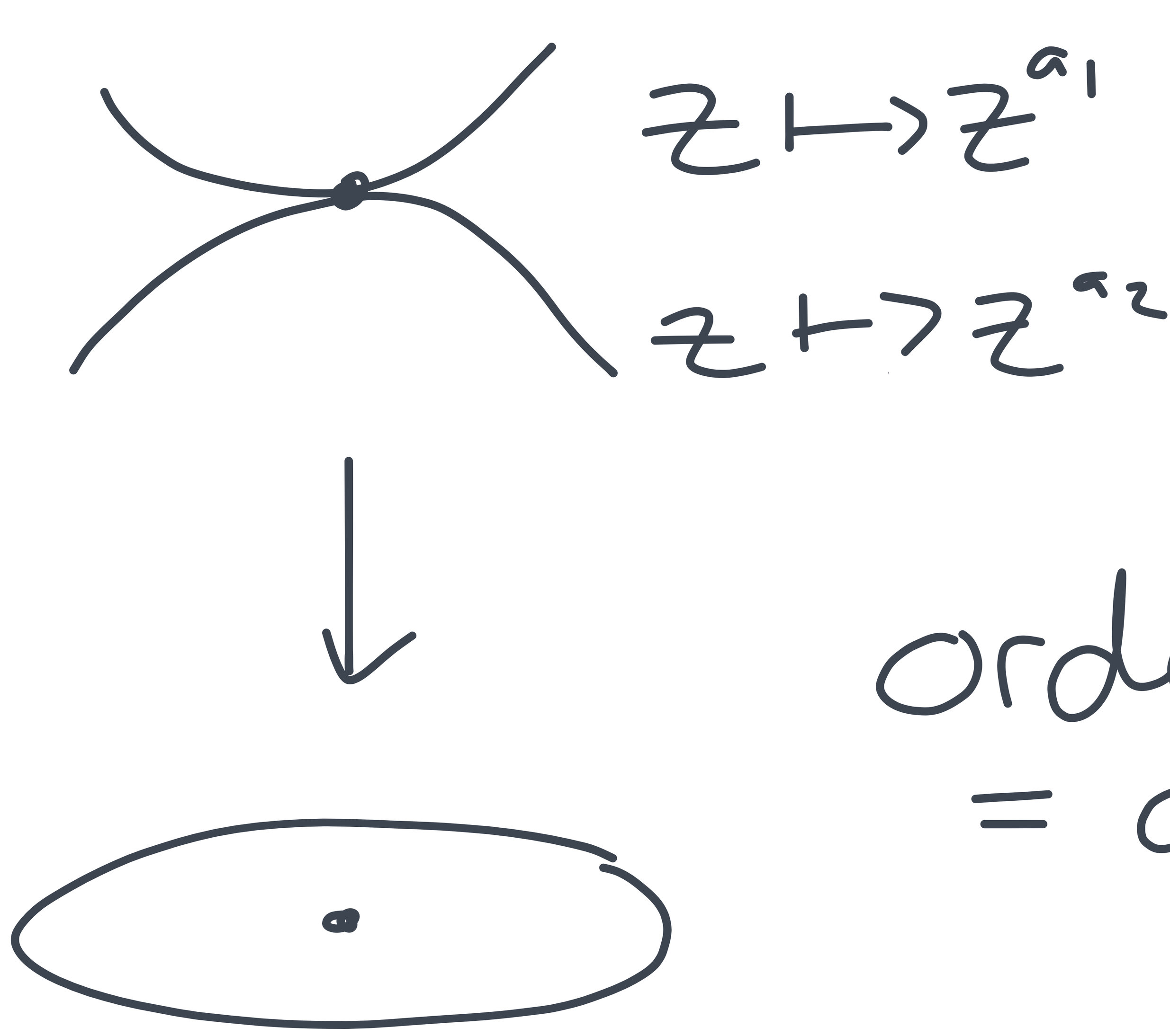
$$\delta: \mathcal{O}_C \longrightarrow \omega_C^{\log} \otimes f^* T_{\mathbb{P}^1}^{\log}$$

← This section "is" the ramification

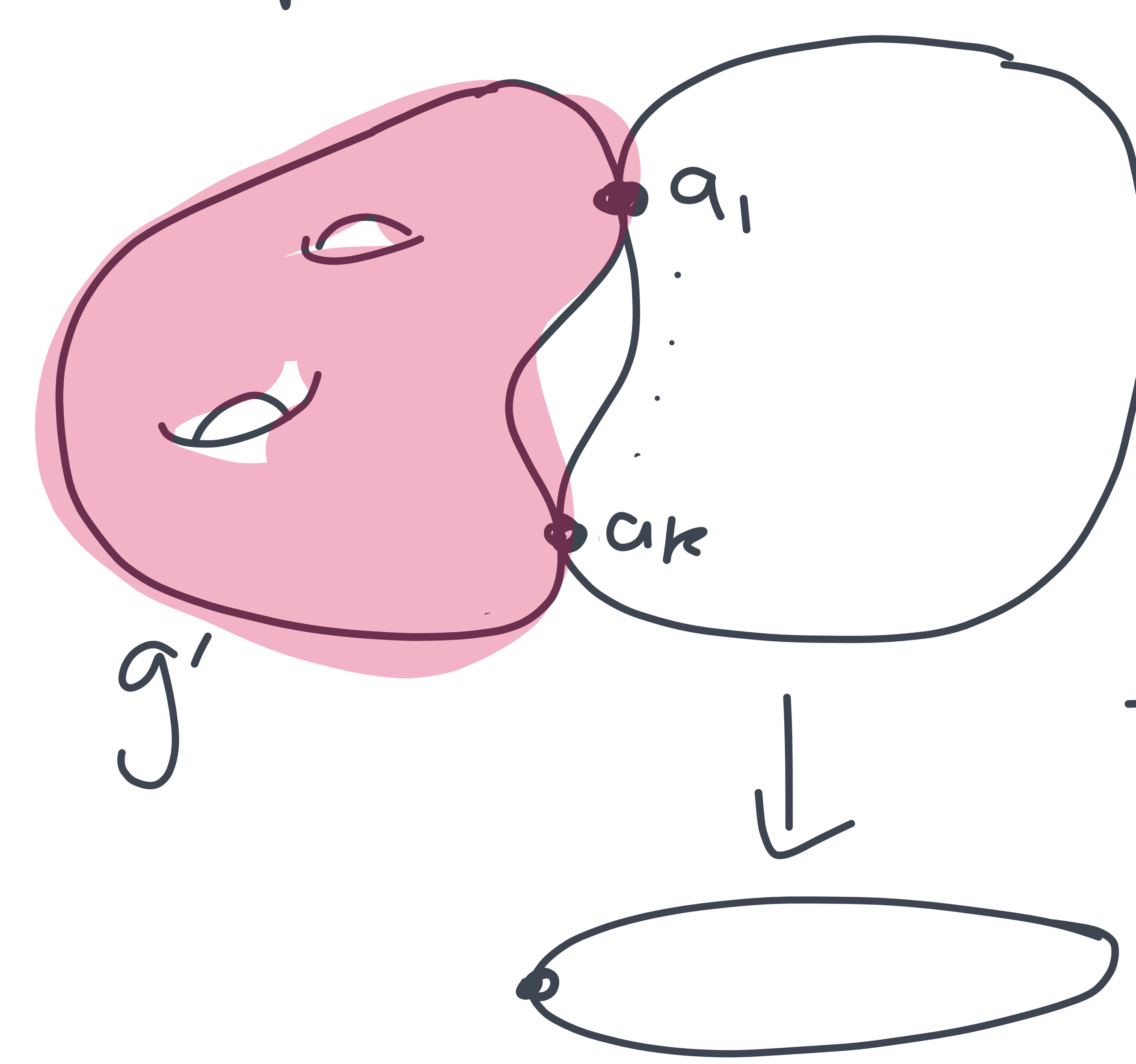
constructed from

i) $df: f^*\Omega_{\mathbb{P}^1} \longrightarrow \Omega_C$ ii) $\Omega_C \longrightarrow \omega_C$

Ramification orders of special loci:



order $= a_1 + a_2$



order $= 2g' - 2 + k + \sum_i a_i$

Want Condition (✡): A moduli space whose generic closed moduli points correspond to the stable maps $[f] \in \mathcal{M}_g(\mathbb{P}^1, \lambda)$ with the property:
- All ramification orders are equal to r.

Easiest to define a space where the "ramification divisor" is "divisible by r":

Define $\overline{\mathcal{M}}_g^{\frac{1}{r}}(\mathbb{P}^1, \lambda)$ (or $\mathcal{M}^{\frac{1}{r}}$) to parametrise:

$$\left(f: C \to \mathbb{P}^1,\ L,\ e: L^r \xrightarrow{\sim} \omega_C^{\log} \otimes f^* T_{\mathbb{P}^1}^{\log},\ \sigma: \mathcal{O}_C \to L\right)$$

with
i) $C \in \mathfrak{M}_g^r$ (nodes are r-orbifold points)
ii) $[\bar{f}: \bar{C} \to \mathbb{P}^1] \in \overline{\mathcal{M}}_g(\mathbb{P}^1, \lambda)$ (coarse curve/map)
iii) L a line bundle and e an isomorphism
iv) $\delta = e(\sigma^r)$ (where δ is the ramification section).

Thm [L.'18] $\mathcal{M}^{\frac{1}{r}}$ is a proper DM stack satisfying contition (☆). Moreover there is $\widehat{br}$ such that:

$$\begin{tikzcd} \mathcal{M}^{\frac{1}{r}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Sym}^m \mathbb{P}^1 \arrow[d] \\ \mathcal{M} \arrow[r] & \mathrm{Sym}^{rm} \mathbb{P}^1 \end{tikzcd}$$

(Note: This is not cartesian.)

Proof (Idea) ① Construct

$$\mathcal{M}^r = \left\{ \left(f: C \to \mathbb{P}^1, L, e: L^r \xrightarrow{\sim} \omega_C^{\log} \otimes f^* T_{\mathbb{P}^1}^{\log} \right) \right\} /_{\sim}$$

Use [Ch'08] to show the forgetful map

$$\mathcal{M}^r \longrightarrow \overline{\mathcal{M}}_g(\mathbb{P}^1, \lambda)$$

is proper, flat, surjective, DM-type of degree r^{2g-1}.

② Use a natural inclusion defined by δ:

$$\begin{tikzcd}
\mathcal{M}^{\frac{1}{r}} \arrow[r, hook] \arrow[d] & \mathrm{Tot}_{\mathcal{M}^r} \pi_* \mathcal{L} \arrow[d] & \sigma \arrow[d, mapsto, "\text{Proper + DM-type}"] \\
\mathcal{M}^r \arrow[r, hook] & \mathrm{Tot}_{\mathcal{M}^r} \pi_* \mathcal{L}^r & \sigma^r \\
(f,L,e) \arrow[r, mapsto] & (f,L,e,e^{-1}(\delta)) &
\end{tikzcd}$$

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③ Cond ($\star$) is shown directly:

i) <u>Node with $z \mapsto z^{a_i}$</u>: $\sigma^r = e^{-1}(\delta)$ gives L^r local generator $N_*(v_1^{-a_1 r} - v_2^{-a_2 r})$ and L has local generator $N_*(v_1^{-a_1} - v_2^{-a_2})$, μ_r-equiv. $\Rightarrow r \mid (a_1+a_2)$

ii) <u>$B \subset C$ contracted</u>: Have $r \mid \deg(\overline{L|_B})$.

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④ Construction of $\hat{br}$ mirrors br:

$$br(f) = \mathrm{div}(Rf_* \delta) \rightsquigarrow \hat{br}(f) = \mathrm{div}(Rf_* \sigma)$$

$\square$

Thm [L. '18 & '20] There are P.O.T.s φ, ϕ, ψ such that if $\nu : \mathcal{M}^{\frac{1}{r}} \to \mathcal{M}^r$ forgets $\sigma : \mathcal{O}_C \to L$ then

$$\begin{tikzcd}
\mathbb{T}_\nu \arrow[r] \arrow[d, "\varphi"] & \mathbb{T}_{\mathcal{M}^{\frac{1}{r}}} \arrow[r] \arrow[d, "\phi"] & \nu^*\mathbb{T}_{\mathcal{M}^r} \arrow[r] \arrow[d, "\nu^*\psi"] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{T}_\nu[1] \arrow[d, "\varphi[1]"] \\
\mathbb{E}_\nu \arrow[r] & \mathbb{E}_{\mathcal{M}^{\frac{1}{r}}} \arrow[r] & \nu^*\mathbb{E}_{\mathcal{M}^r} \arrow[r] & \mathbb{E}[1]
\end{tikzcd}$$

has distinguished triangles as rows,

$$\mathbb{E}_\nu \cong R\pi_*\left[\mathcal{L} \xrightarrow{\sigma^{r-1}} \mathcal{L}^r\right]$$

Proof (Idea) $\mathbb{E}_\nu$ constructed from

$$\begin{tikzcd}
\mathcal{M}^{\frac{1}{r}} \arrow[d, "\nu"'] & \mathcal{C}^{\frac{1}{r}} \arrow[l] \arrow[r, hook, "s^*\sigma"'] \arrow[d] & \mathrm{Tot}_{\mathcal{C}^r}\mathcal{L} \arrow[r, "\beta"'] \arrow[d, "\tau"] & \mathcal{C}^r \arrow[d, equal] \\
\mathcal{M}^r & \mathcal{C}^r \arrow[l] \arrow[r, hook, "s^*e^{-1}(\delta)"'] & \mathrm{Tot}_{\mathcal{C}^r}\mathcal{L}^r \arrow[r, "\alpha"'] & \mathcal{C}^r
\end{tikzcd}$$

- $\tau : \xi \mapsto \xi^r$
- $\mathbb{T}_\alpha \cong \alpha^*\mathcal{L}^r$
- $\mathbb{T}_\beta \cong \beta^*\mathcal{L}$
- Need to show connecting morphism exists.

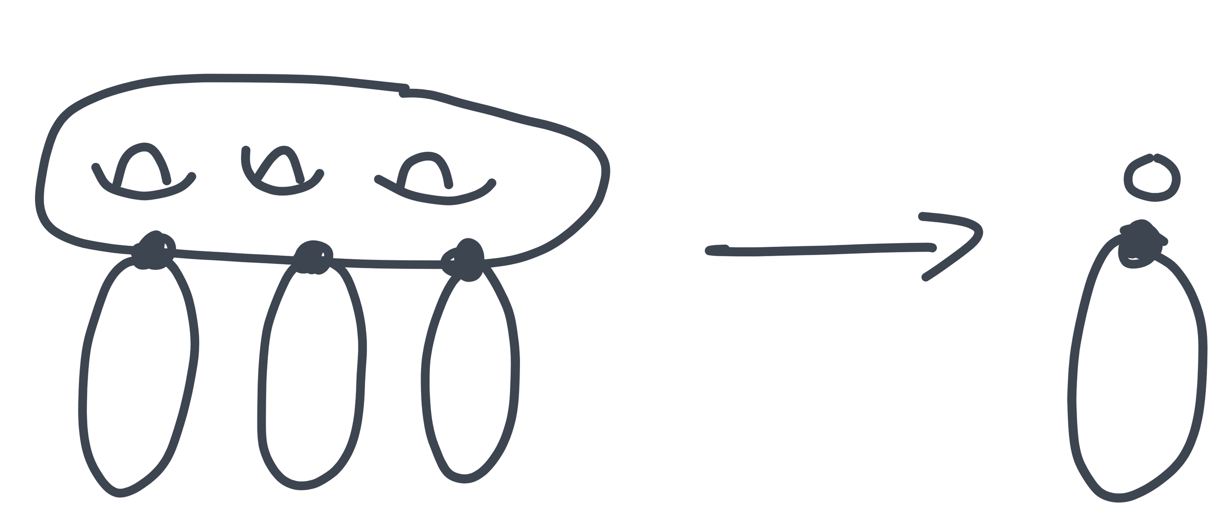
$\square$

Thm [L.'20]

$$\int_{[\overline{\mathcal{M}}^{\frac{1}{r}}_{g}(\mathbb{P}^1,\lambda)]^{vir}} \widehat{br}^*[p_1+\cdots+p_m] = r^{m+r+2g-2}\, m! \left(\prod_i \frac{\left(\frac{\lambda_i}{r}\right)^{\lfloor\frac{\lambda_i}{r}\rfloor}}{\lfloor\frac{\lambda_i}{r}\rfloor!}\right) \int_{\overline{\mathcal{M}}^{\frac{1}{r}}_{g,n}} \frac{c(-R\pi_*\mathcal{L})}{\prod_i\left(1-\frac{\lambda_i}{r}\psi_i\right)}$$

<u>Proof</u> (Idea) Use virtual localisation as in [FP'02]:

- Choose an equivariant lift of $\gamma = [p_1+\cdots+p_m]$ such $\widehat{br}^*\gamma$ vanishes on all the fix loci except

$$F := \Bigg\{$$  $$\Bigg\}$$

$$\rightsquigarrow \quad \int_{[\mathcal{M}^{\frac{1}{r}}]} \widehat{br}^*\gamma = \int_{[F]^{vir}} \frac{\widehat{br}^*\gamma}{c_{top}(N_F^{vir})}$$

$[F]^{vir}$ from $\left(\mathbb{E}_{\mathcal{M}^{\frac{1}{r}}}|_F\right)^{fix}$, $N_F^{fix} = \left(\mathbb{E}_{\mathcal{M}^{\frac{1}{r}}}|_F\right)^{fix}$

3 contributions to fixed/moving parts:

a) $\mathbb{E}_{\mathcal{M}^r} \longrightarrow$ "fix" is concentrated in degree 0 & is locally free

b) $R\pi_* \mathcal{L}^r \cong R\pi_*(\omega_\pi^{\log} \otimes f^* T_{\mathbb{P}^1}^{\log})$

c) $R\pi_* \mathcal{L}$

$\Big\}\ \mathbb{E}_\nu \longrightarrow$ Has "fix" $= 0$.

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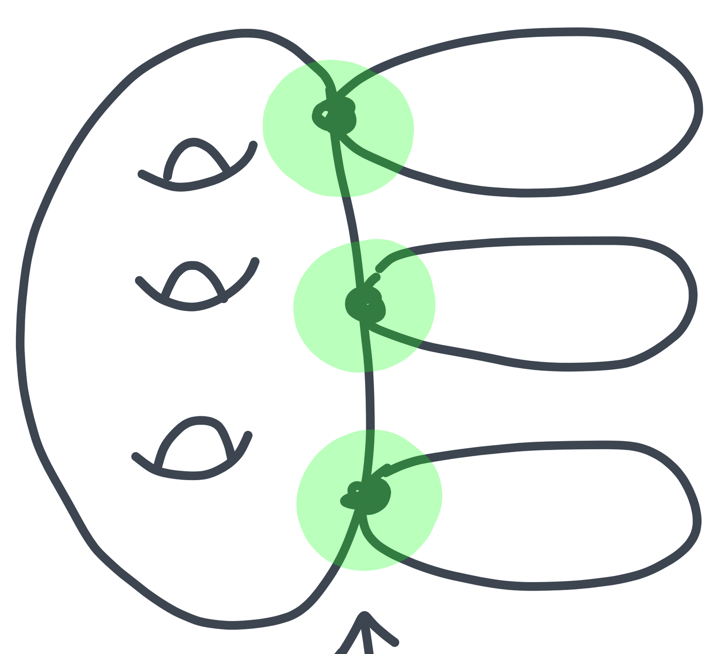
a) Same analysis as for $\overline{\mathcal{M}}_g(\mathbb{P}^1, \lambda)$ except



$= C$

r-orbifold pts.

Def_C is an r^ℓ-fold branched cover of $\mathrm{Def}_{\overline{C}}$. Also,

$$\boxed{\lambda_i \psi_i} \rightsquigarrow \boxed{\tfrac{1}{r}\lambda_i \psi_i}$$

b) Cancels $\left(\prod \frac{\lambda_i^{\lambda_i}}{\lambda_i}\right) \cdot c(-\pi_* \omega_\pi)$ term from a).

c) Gives $\left(\prod \frac{(\,)^{\lfloor\ \rfloor}}{\lfloor\ \rfloor !}\right) \cdot c(-R\pi_* \mathcal{L})$ factor.

"fix" properties give $[F]^{vir} = [F]$. Also have map $F \to \overline{\mathcal{M}}_{g,\ell}^{\frac{1}{r}}$ of degree $(\lambda_1 \cdots \lambda_\ell)^{-1}$ and the contributions are pulled back by this map. $\rightsquigarrow$ Can use projection formula. $\square$

"r"-Hurwitz Relationships

$$\int_{[\overline{\mathcal{M}}^{\frac{1}{r}}_{g}(\mathbb{P}^1,\lambda)]^{vir}} \widehat{br}^*[p_1+\cdots+p_m] \xleftrightarrow{[L.20]} r^{\ell+m+2g-2}\, m! \left(\prod_i \frac{\left(\frac{\lambda_i}{r}\right)^{\lfloor \frac{\lambda_i}{r}\rfloor}}{\lfloor \frac{\lambda_i}{r}\rfloor !}\right) \int_{\overline{\mathcal{M}}^{\frac{1}{r}}_{g,\ell}} \frac{c(-R\pi_*\mathcal{L})}{\prod_i (1-\frac{1}{r}\lambda_i\psi_i)}$$

In progress

!!

$$H^r_{g,\lambda}$$

$$\int_{[\overline{\mathcal{M}}_{g,m}(\mathbb{P}^1,\lambda)]^{vir}} \prod_i \psi_i^r\, ev_i^*[p_i]$$

Topological recursion

Rep. theory of $Z(\mathbb{C}[S_d])$

Factorisations in S_d

In progress:

$$\int_{[\mathcal{M}^{\frac{1}{r}}]^{vir}} \hat{br}^*[p_1 + \cdots + p_m] = (r!)^m \int_{[\overline{\mathcal{M}}_{g,m}(\mathbb{P}^1,\lambda)]^{vir}} \prod_i \psi_i^r ev_i^*[p_i]$$

Prove in two parts:

① Consider the 1-point case

② Use degeneration from [Li '02]